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STONE-ČECH COMPACTIFICATION OF LOCALES II

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Introduction

The existence of the Stone-Čech compactification of a topological space is equivalent, classically, to the Prime Ideal Theorem, and hence only slightly weaker than the Axiom of Choice. In an earlier paper, we proved *constructively* the existence of the Stone-Čech compactification of a locale by lattice theoretic arguments based on [1]. Independently, Johnstone [6] established the same result by a constructive modification of the method of Tychonoff [10].

Among the many other ways of obtaining the Stone-Čech compactification of a space, the most significant one is that which describes it as the space of maximal ideals of its algebra of bounded continuous real-valued functions. The present paper presents the constructive analogue of this approach, based on a syntactic description of the locale of maximal ideals of this algebra introduced in [9].

1. The locale Max $\mathbb{R}(L)$

Recall that a *locale* L is a complete lattice in which

$$a \wedge \bigvee S = \bigvee a \wedge b \qquad (b \in S)$$

for any $a \in L$ and $S \subseteq L$. That a *map of locales*

$$f : L \to L'$$

is a mapping f^* from L' to L preserving finite meets and arbitrary joins: the $\wedge\bigvee$ homomorphism f^* of complete lattices will be called the *inverse image mapping* of the map f of locales. In particular, consider [8] the *locale* $\mathbb{R}$ *of reals:* that is, the locale generated by the rational open intervals (a, b), in which a, b denote rational numbers, together with relations given by:

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(i) $(a,b)=0$ when $a \geq b$,

(ii) $(a,b) \wedge (c,d) = (a \vee c, b \wedge d)$,

(iii) $(a,b) \vee (c,d) = (a,d)$ when $a \leq c < b \leq d$,

(iv) $(a,b) = \bigvee_{a<c<d<b} (c,d)$,

(v) $1 = \bigvee_{a<b} (a,b)$.

A map of locales from L to $\mathbb{R}$ will be called a *continuous real function* on the locale L: such a function will be said to be *bounded* provided that there is some rational open interval (a,b) of which the inverse image is the unit of the locale L. The bounded continuous real functions on L inherit from the locale $\mathbb{R}$ the structure of a lattice-ordered ring, which will be denoted by $\mathbb{R}(L)$. For an introduction to the theory of locales, of which these ideas form part, the reader is referred to [7]. The locale of reals, introduced originally by Joyal [8], is considered in [5], to which the ideas exposed in [4] also relate.

It will be proved that for any locale L there is a locale Max $\mathbb{R}(L)$, coinciding classically with the locale of the maximal ideal space of the ring $\mathbb{R}(L)$, together with a map

$$L \to \mathrm{Max}\, \mathbb{R}(L)$$

of locales giving the Stone–Čech compactification of L. The description of Max $\mathbb{R}(L)$, and the proof of its universality, will be constructive, allowing the result to be internalised in any intuitionistic context. It is this consideration, among others, which leads to the need to work with locales rather than topological spaces: without some form of choice Max $\mathbb{R}(L)$ may not be the locale of a topological space. Nevertheless, Max $\mathbb{R}(L)$, in view of the role which it plays, will be called the *locale of maximal ideals* of the ring $\mathbb{R}(L)$.

The locale Max $\mathbb{R}(L)$ is constructed in the following way: consider the propositional theory obtained from the ring $\mathbb{R}(L)$ by introducing a proposition

$$a \in A(q)$$

for each $a \in \mathbb{R}(L)$ and each nonnegative rational number q, together with the following axioms:

(A1) $\mathit{true} \vdash 1 \in A(r)$ when $r < 1$,

(A2) $a \in A(r) \vdash \mathit{false}$ when $|a| \leq r$,

(A3) $a + b \in A(r+s) \vdash a \in A(r) \vee b \in A(s)$,

(A4) $ab \in A(rs) \vdash a \in A(r) \vee b \in A(s)$,

(A5) $a \in A(r) \wedge b \in A(s) \vdash a^2 + b^2 \in A(r^2 + s^2)$, (A6) $a \in A(r) \wedge b \in A(s) \vdash ab \in A(rs)$,

(A7) $a \in A(r) \dashv\vdash \bigvee_{r<s} a \in A(s)$.

The intention of the axiomatisation is that the proposition $a \in A(q)$ expresses the extent to which the continuous function a has absolute value greater that q at a point of the Stone-Čech compactification. The language considered is that which allows only the propositional connectives $\wedge$, $\bigvee$, *true* and *false*. The rules of intuitionistic propositional logic give the notion of provability in the theory. The locale Max $\mathbb{R}(L)$ is then that of propositions in the theory modulo provable equivalence, with the ordering given by provable entailment. (Cf. [9].)

The locale Max $\mathbb{R}(L)$ is definable constructively: the form of the language and the rules of deduction mean that it is precisely the locale generated by the symbols $a \in A(q)$ subject to relations expressed by the axioms of the theory. In consequence of this, a map from a locale M to the locale Max $\mathbb{R}(L)$ is exactly a model of the theory in the locale M: that is, assigns to each proposition $a \in A(q)$ an element of M in such a way that the axioms, $\phi \vdash \psi$, of the theory are validated, $\phi \models \psi$, in the locale M. With the canonical interpretation of the proposition $a \in A(q)$ in mind, one particular map which may be defined is that from the locale L given by assigning to each proposition $a \in A(q)$ the interpretation in the locale L of the proposition

$$|a| > q$$

in the language of sheaves on L obtained by identifying a with a section of the sheaf $\mathbb{R}_L$ of continuous real functions on the locale L. That each axiom of the theory is validated in this interpretation may readily be verified. One therefore obtains a canonical map

$$L \to \text{Max}\, \mathbb{R}(L),$$

concerning which will be established the following:

Proposition. *For any locale L, the canonical map*

$$L \to \text{Max}\, \mathbb{R}(L)$$

is universal among maps to compact completely regular locales.

The map will therefore be called the *Stone-Čech compactification* of the locale L. It may be recalled that a locale M is said to be *compact* provided that

$$1 = \bigvee S \quad \text{implies} \quad 1 \in S$$

for any up-directed subset $S \subseteq M$. The locale M is said to be *regular* provided that each $b \in M$ is the join of those $a \in M$ which are rather below it: $a \in M$ is said to be *rather below* $b \in M$, written $a \lhd b$, provided that there exists some $c \in M$ for which

$$a \wedge c = 0 \quad \text{and} \quad c \vee b = 1.$$

The locale M is said to be *completely regular* provided that each $b\in M$ is the join of those $a\in M$ which are completely below it: $a\in M$ is said to be *completely below* $b\in M$, written $a\lll b$, provided that there exists an interpolation $d_{ik}\in M$, for $i=0,1,\dots$ and $k=0,1,\dots,2^i$ dependent on i, such that

(i) $d_{00}=a$ and $d_{01}=b$,

(ii) $d_{ik}\lhd d_{i\,k+1}$,

(iii) $d_{ik}=d_{i+1\,2k}$

for all appropriate i,k.

The existence of this Stone–Čech compactification was established in an earlier article [2]. Independently, although by rather different means, its existence was observed in [6]. Each of these papers also proved that a compactification existed for the case of compact regular locales. Provided that dependent choice is assumed, the concepts of regularity and complete regularity coincide for compact locales: the rather below relation may be shown to interpolate, agreeing therefore with the completely below relation. Otherwise these concepts give apparently different interpretations of the classical concept of a compact Hausdorff space, the corresponding compactifications being therefore distinct. In the presence of choice, each concept is actually equivalent to that of a compact Hausdorff space. Moreover, for any locale L, the points of Max $\mathbb{R}(L)$ may then be identified with the maximal ideals of the ring $\mathbb{R}(L)$: to each point of the locale Max $\mathbb{R}(L)$ there corresponds the maximal ideal of $\mathbb{R}(L)$ consisting of those elements a for which the zero of the locale $\mathbb{I}$ is assigned to each proposition $a\in A(q)$. To this extent the construction generalises the classical one of the Stone–Čech compactification.

The proof of the existence of the compactification to be given here seems to be of a rather different nature to those obtained elsewhere, whether by considering the locale of completely regular ideals of L [2] or by applying the adjoint functor theorem after proving Tychonoff's theorem [6]. It differs also from the classical proof for a topological space in terms of the maximal ideal space of the ring of continuous functions, both in depending only on elementary properties of the reals and in being almost entirely syntactic. The technique will be to prove that the canonical interpretation

$$L\to \text{Max}\ \mathbb{R}(L)$$

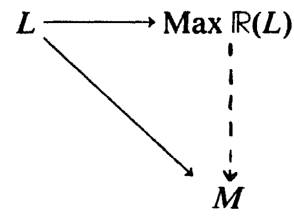
of the theory is complete precisely if the locale L is compact and completely regular. It will be shown first that if L is compact and completely regular then the canonical map is an isomorphism of locales. The result then follows on showing that Max $\mathbb{R}(L)$ is compact and completely regular for any locale L.

2. The universality of Max $\mathbb{R}(L)$

It will be proved that the canonical map from a locale L to the locale Max $\mathbb{R}(L)$

of maximal ideals of the ring $\mathbb{R}(L)$ of bounded continuous real functions on L has the following property:

(2.1) *For each map $L \to M$ from L to a compact completely regular locale M, there exists a unique map of locales from* Max $\mathbb{R}(L)$ *to M such that the diagram*



commutes.

Before commencing the proof, it will be helpful to make some preliminary remarks concerning the theory of Max $\mathbb{R}(L)$. These are motivated by the observation that the maximal ideals of $\mathbb{R}(L)$ are classically exactly the prime ideals which are closed. The conclusion will allow the theory of the locale Max $\mathbb{R}(L)$ to be described rather more simply than in terms of the propositions

$$a \in A(q).$$

For the lattice operations of the ring $\mathbb{R}(L)$ allow the *absolute value* of any element to be defined by:

$$|a| = a \vee -a.$$

An immediate observation is that:

$$a \in A(q) \dashv\vdash |a| \in A(q), \tag{2.2}$$

leaving only the nonnegative elements of $\mathbb{R}(L)$ to be considered. For one has that $a \in A(q) \vdash a^2 \in A(q^2) \vdash |a|^2 \in A(q^2) \vdash |a| \in A(q)$, respectively by (A6), the equality $a^2 = |a|^2$, and (A4). The converse is proved similarly. From this one obtains:

$$a \in A(r) \wedge b \in A(s) \vdash |a| + |b| \in A(r+s). \tag{2.3}$$

For there exist $c, d \in \mathbb{R}(L)$ with $c^2 = |a|$, $d^2 = |b|$. Then $a \in A(r) \vdash \bigvee_{r'>r} a \in A(r')$. Choose $r' > r$ with $a \in A(r')$, then p such that $r < p^2 < r'$. Then $a \in A(p^2)$, hence $|a| \in A(p^2)$ by (2.2). Similarly, $|b| \in A(q^2)$ for some q with $q^2 > s$. It follows that $c \in A(p)$, $d \in A(q)$ by (A4). Hence, $|a| + |b| \in A(p^2 + q^2)$ by (A5), giving that $|a| + |b| \in A(r+s)$ by (A7). From this in turn one has:

$$a \in A(r) \vdash |a| + q \in A(r+q) \tag{2.4}$$

for any positive rational q. For if $a \in A(r)$, choose $r' > r$ with $a \in A(r')$. Then find $q' < q$ with $r + q < r' + q'$. Then $q \in A(q')$ by (A1), (A2) and (A4). So $a \in A(r') \wedge q \in A(q') \vdash |a| + q \in A(r' + q')$, by (2.3), hence $|a| + q \in A(r+q)$ since $r + q < r' + q'$, by (A7).

The lattice structure of $\mathbb{R}(L)$ also allows one to define the *cutdown*

$$(a:q)=(|a|-q)\vee 0$$

of any $a\in\mathbb{R}(L)$ by any positive rational q: note that this depends only on the absolute value of $a\in\mathbb{R}(L)$. These cutdown elements allow the description of Max $\mathbb{R}(L)$ to be re-examined, starting with the observation that:

$$a\in A(q)\dashv\vdash(a:q)\in A(0), \tag{2.5}$$

expressing the intuitive idea that the extent to which a has absolute value greater than q should be that to which its cutdown by q is positive.

Before proving the equivalence, observe that writing

$$(q:a)=(a:q)+q-|a|,$$

one has that

$$(a:q)=(|a|-q)^+ \quad\text{and}\quad (q:a)=(|a|-q)^-,$$

from which it follows that $(a:q)(q:a)$ is always zero.

Now suppose that $(a:q)\in A(0)$. Then choose $r>0$ such that $(a:q)\in A(r)$, by (A7). Then

$$\begin{aligned}(a:q)\in A(r)&\vdash(a:q)+q\in A(r+q) &&\text{by (2.4)}\\ &\vdash(q:a)+|a|\in A(r+q)\\ &\vdash(q:a)\in A(r)\vee|a|\in A(q) &&\text{by (A3)}\\ &\vdash|a|\in A(q),\end{aligned}$$

since $(a:q)(q:a)=0$ implies $(q:a)\in A(r)\vdash$ *false* by (A2) and (A6). But then,

$$\vdash a\in A(q) \qquad \text{by (2.2).}$$

Conversely, suppose $a\in A(q)$. Then:

$$\begin{aligned}a\in A(q)&\vdash|a|\in A(q) &&\text{by (2.2)}\\ &\vdash|a|-(a:q)\in A(q)\vee(a:q)\in A(0) &&\text{by (A3)}\\ &\vdash(a:q)\in A(0)\end{aligned}$$

since $0\leq(q:a)\leq q$ implies $0\leq q-(q:a)\leq q$ implies $0\leq|a|-(a:q)\leq q$ implies $|a|-(a:q)\in A(q)\vdash$ *false*, by (A2), which completes the proof.

This equivalence allows the presentation of Max $\mathbb{R}(L)$ to be simplified by introducing the expression

$$a\in P$$

to denote the proposition $a\in A(0)$ henceforth: the abbreviation is intended to recall the fact that classically every maximal ideal is prime. Indeed, the observation which may mow be made is that:

(2.6) *The propositions $a \in P$ satisfy the following conditions:*

(P1) $true \vdash 1 \in P$,

(P2) $0 \in P \vdash false$,

(P3) $a+b \in P \vdash a \in P \vee b \in P$,

(P4) $ab \in P \dashv\vdash a \in P \wedge b \in P$.

It will be recalled that the theory which these axioms describe is that of the locale Spec $\mathbb{R}(L)$ of prime ideals in the ring $\mathbb{R}(L)$. That these axioms are satisfied by the propositions $a \in P$ of the theory of Max $\mathbb{R}(L)$ follows almost immediately from the axioms of that theory: (P1) and (P2) follow from (A1) and (A2) in each case by taking $r=0$. And (P3) from (A3), and one direction of (P4) from (A6), by taking $r=s=0$. For the converse of (P4), suppose $ab \in P$. Then applying (A4) with $r=0$ and s an upper bound on the function $|b|$, one has $ab \in A(0) \vdash a \in A(0) \vee b \in A(s)$. But $b \in A(s) \vdash false$, by (A2). Hence $ab \in P \vdash a \in P$. Similarly, $ab \in P \vdash b \in P$, which completes the proof.

Finally, it may be noted that in the present situation the propositions satisfy the additional condition:

(P5) $a \in P \vee b \in P \vdash a^2+b^2 \in P$

which allows any finite disjunction of propositions of this form to be replaced by a single one. For if $a \in P$, then for some $r>0$, $a \in A(r)$. But

$$a \in A(r) \vdash a^2 \in A(r^2) \qquad \text{by (A6)}$$

$$\vdash a^2+b^2 \in A(r^2-s^2) \vee b^2 \in A(s^2) \qquad \text{by (A3) where } s<r$$

$$\vdash a^2+b^2 \in P \vee b \in A(s) \qquad \text{by (A4).}$$

But

$$a \in A(r) \wedge b \in A(s) \vdash a^2+b^2 \in A(r^2+s^2) \qquad \text{by (A5)}$$

$$\vdash a^2+b^2 \in P \qquad \text{by (A7).}$$

So $a \in A(r) \vdash a^2+b^2 \in P$. Hence $a \in P \vdash a^2+b^2 \in P$, and similarly $b \in P \vdash a^2+b^2 \in P$. It may be noted that in the course of the proof it has also been shown that:

$$a \in A(r) \vee b \in A(r) \vdash a^2+b^2 \in A(r^2-s^2) \qquad (2.7)$$

for any $0<s<r$.

The main consequence of these observations will be examined later. For the moment, one obtains the following fact which allows the locale Max $\mathbb{R}(L)$ to be described rather more succinctly:

(2.8) *Any proposition ψ of the theory* Max $\mathbb{R}(L)$ *is provably equivalent to the disjunction*

$$\bigvee a \in P \; (a \in P \vdash \psi)$$

of the propositions $a \in P$ which provably entail ψ.

For any $b \in A(q)$ is equivalent to some $a \in P$ (by (2.5)). Any finite conjunction $a_1 \in P \wedge \cdots \wedge a_n \in P$ of such propositions is provably equivalent to $a_1 \cdots a_n \in P$ (by (P4)). And the propositions of the theory are exactly arbitrary disjunctions of finite conjunctions of propositions of the form $b \in A(q)$. It may be added that because of (P5) the disjunction appearing in the equivalence of (2.8) is always over an up-directed set of propositions.

With these preliminary remarks, it may be proved that:

(2.9) *If M is a compact completely regular locale, then the canonical map*

$$M \to \operatorname{Max} \mathbb{R}(M)$$

is an isomorphism of locales.

The proof of this, resting on the description of the propositions of $\operatorname{Max} \mathbb{R}(M)$ given by (2.8), provides the main part of the proof of (2.1). To prove it, recall that the canonical map is defined by requiring the inverse image of each $a \in A(q)$ of $\operatorname{Max} \mathbb{R}(M)$ to be the interpretation of the proposition $|a| > q$ in the sheaf of continuous real functions on the locale M. It is enought therefore to prove that every element of M is the join of elements of this form, and that the inverse image mapping preserves and reflects the order relations of the locales. The former may be verified immediately from the complete regularity of M, while the latter requires it to be proved that:

(2.10) $\phi \vdash \psi$ *if and only if* $\phi \models \psi$,

for any propositions ϕ, ψ of the theory of $\operatorname{Max} \mathbb{R}(M)$, in which $\vdash$ denotes provable entailment in the theory and $\models$ entailment in the interpretation in the locale M. Applying the remark (2.8) concerning the theory $\operatorname{Max} \mathbb{R}(M)$, it is enough to show that:

(2.11) $a \in P \vdash \psi$ *if and only if* $a \in P \models \psi$,

for any element $a \in \mathbb{R}(M)$ and any proposition ψ of the theory $\operatorname{Max} \mathbb{R}(M)$.

To prove this, note that the forward implication is immediate. Conversely, the proof uses the fact that Urysohn's lemma holds for a compact completely regular locale to manipulate the propositions of the theory, together with the compactness of the locale to allow certain disjunctions to be made finite. Suppose then that $a \in P \models \psi$. Assert that for each n there exists $c_n \in \mathbb{R}(M)$ such that:

(i) $\models a \in P \vee c_n = 0$ and $\models \neg a \in A(1/n) \vee c_n = 1$,

(ii) $a \in P \dashv\vdash \bigvee_n a_n \in P$ where $a_n = ac_n$.

For each n the existence of c_n follows from the fact that $0<|a| \vee |a|<1/n$. For Urysohn's lemma then gives $c_n \in \mathbb{R}(M)$ with $0 \le c_n < 1$ satisfying $0<|a| \vee c_n = 0$ and $|a|<1/n \vee c_n = 1$ over M. Then (i) follows: the first part since $0<|a|$ interprets $a \in P$, and the second sicne $|a|<1/n \rightarrow |a| \le 1/n$, which interprets $\neg a \in A(1/n)$. Now let $a_n = ac_n$. For any n, $|a|<1/n \vee c_n = 1$. If $|a|<1/n$, then $|a(1-c_n)|<1/n$ since $0 \le c_n \le 1$. While if $c_n = 1$, then $|a(1-c_n)|$ equals zero. Hence, for each n, $|a-a_n|< 1/n$. So since (by (A2))

$$a - a_{2n} \in A(1/2n) \vdash \mathit{false}$$

and (by (A3))

$$a \in A(1/n) \vdash a - a_{2n} \in A(1/2n) \vee a_{2n} \in A(1/2n),$$

it follows that $a \in A(1/n) \vdash a_{2n} \in A(1/2n)$. Thus $a \in A(1/n) \vdash \bigvee_m a_m \in P$ (by (A7)). But $a \in P \vdash \bigvee_n a \in A(1/n)$ (by (A7)). Thus $a \in P \vdash \bigvee_m a_m \in P$, and the converse is immediate.

Assert now that for each n there exists $d_n \in \mathbb{R}(M)$ and $q_n > 0$ such that:

(iii) $\quad \vDash c_n = 0 \vee d_n \in A(q_n)$, and

(iv) $\quad d_n \in P \vdash \psi$.

For given that $a \in P \vDash \psi$ one has $\vDash c_n = 0 \vee \psi$ for each n (by (i)). Expressing ψ in the form $\bigvee b \in P\ (b \in P \vdash \psi)$, noting that $b \in P \vdash \bigvee_{q>0} b \in A(q)$ and using the compactness oi M, the disjunction may be made finite for each n in such a way that:

$$\vDash c_n = 0 \vee b_1 \in A(p_n) \vee \cdots \vee b_m \in A(p_n)$$

for some $p_n > 0$ and $b_1, \ldots, b_m \in \mathbb{R}(M)$ for which $b_i \in P \vdash \psi$ for $i = 1, \ldots, m$ depending on n. Applying (2.7) and letting $d_n = b_1^2 + \cdots + b_m^2$, there is $q_n < p_n$ such that $b_1 \in A(p_n) \vee \cdots \vee b_m \in A(p_n) \vdash d_n \in A(q_n)$. In $d_n \in P \vdash b_1 \in P \vee \cdots \vee b_m \in P$ (by (P3)) and $b_i \in P \vdash \psi$ for each i.

Now, finally, $a \in P \vdash \psi$. For one already has that $a \in P \vdash \bigvee_n a_n \in P$, and the above shows that $\bigvee_n d_n \in P \vdash \psi$. The required entailment is therefore proved by establishing that $a_n \in P \vdash d_n \in P$ for each n. But, for each n, since $\vDash c_n = 0 \vee d_n \in A(q_n)$, one may define

$$e_n = \begin{cases} c_n/d_n & \text{where } d_n \in A(q_n), \\ 0 & \text{where } c_n = 0 \end{cases}$$

in the sheaf $\mathbb{R}_M$. This gives an element of $\mathbb{R}(M)$, since $c_n \in \mathbb{R}(M)$ is bounded and $d_n \in \mathbb{R}(M)$ is bounded away from zero by q_n where $d_n \in A(q_n)$. Then, $c_n = d_n e_n$. Hence, $a_n \in P \vdash c_n \in P \vdash d_n \in P$, as required, by (P4). Thus, $a \in P \vdash \psi$, which completes the proof.

It follows that the map

$$M \rightarrow \operatorname{Max} \mathbb{R}(M)$$

is an isomorphism provided that M is a compact completely regular locale.

Finally, suppose given a map of locales

$$f : L \to M.$$

It determines a homomorphism

$$\mathbb{R}(f) : \mathbb{R}(M) \to \mathbb{R}(L)$$

by assigning to each $a \in \mathbb{R}(M)$ the continuous function $af \in \mathbb{R}(L)$ obtained by composing with the map f. The assignment respects the boundedness of the functions considered: explicitly, $|a| < q$ implies $|af| \leq q$ for any nonnegative rational q. Assigning to each proposition $a \in A(q)$ of Max $\mathbb{R}(M)$ the proposition $af \in A(q)$ of Max $\mathbb{R}(L)$, one obtains a map

$$\text{Max}\ \mathbb{R}(f) : \text{Max}\ \mathbb{R}(L) \to \text{Max}\ \mathbb{R}(M)$$

of these locales: that the assignment verifies the axioms of the theory of Max $\mathbb{R}(M)$ follows immediately from the form of the axioms and the remark made above. Moreover, the canonical map from any locale to the locale of maximal ideals of its ring of bounded continuous real functions makes the diagram

$$\begin{array}{ccc} L & \longrightarrow & \text{Max}\ \mathbb{R}(L) \\ \downarrow & & \downarrow \\ M & \longrightarrow & \text{Max}\ \mathbb{R}(M) \end{array}$$

commute. In the particular case when the locale M is taken to be compact and completely regular, the fact that the canonical map from M to Max $\mathbb{R}(M)$ is an isomorphism yields the existence for each map

$$L \to M$$

of a map

$$\text{Max}\ \mathbb{R}(L) \to M$$

for which the diagram

$$\begin{array}{ccc} L & \longrightarrow & \text{Max}\ \mathbb{R}(L) \\ & \searrow & \downarrow \\ & & M \end{array}$$

is commutative.

The uniqueness of this map, establishing the assertion of (2.1), is obtained from the observations that the map

$$L \to \text{Max}\ \mathbb{R}(L)$$

is *dense*, in the sense that the inverse image of an element of Max $\mathbb{R}(L)$ is the zero of L only to the extent that the element is the zero of Max $\mathbb{R}(L)$, and the locale M is *regular*, which it will be recalled means that each element is the join of those rather below it. The regularity of M is evident from the assumption of its complete regularity. The denseness of the canonical map comes from the remark that if the inverse image, which is given by $[\![|a|>0]\!]$, of the proposition $a \in P$ of Max $\mathbb{R}(L)$ is the zero of the locale L, then a is necessarily the zero of the ring $\mathbb{R}(L)$. Hence, $a \in P$ is the zero of the locale Max $\mathbb{R}(L)$, by (P2). Since the propositions $a \in P$ generate the locale, the required condition is satisfied.

But, given maps of locales

$$A \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

of which h is dense and equalises the maps f, g into a regular locale C, note that each $c \in C$ may be expressed in the form $\bigvee a \ (a \in C \mid a \lhd c)$. Recall that $a \lhd c$ means that there exists $b \in C$ such that $a \wedge b = 0$ and $b \vee c = 1$. Then

$$h^*(f^*a \wedge g^*b) = h^*f^*a \wedge h^*g^*b = h^*f^*(a \wedge b) = 0$$

since h equalises f, g and $a \wedge b = 0$. From the denseness of h, it follows that $f^*a \wedge g^*b = 0$. Then $b \vee c = 1$ implies $g^*b \vee g^*c = 1$ implies

$$f^*a = f^*a \wedge (g^*b \vee g^*c) = (f^*a \wedge g^*b) \vee (f^*a \wedge g^*c) = f^*a \wedge g^*c.$$

Hence, $f^*a \leq g^*c$. But $c = \bigvee a \ (a \in C \mid a \lhd c)$ and f^* preserves arbitrary joins, so $f^*c \leq g^*c$. Similarly, $g^*c \leq f^*c$, whence the equality of the maps f, g, which completes the proof.

3. The compactness of Max $\mathbb{R}(L)$

It will now be proved that:

(3.1) *For any locale L, the locale* Max $\mathbb{R}(L)$ *is compact and completely regular.*

Once again, the proof is largely syntactic: it has already been shown that Max $\mathbb{R}(L)$ contains propositions $a \in P$ which may be identified with those generating the theory of the locale Spec $\mathbb{R}(L)$ of prime ideals of the ring $\mathbb{R}(L)$. Moreover, that the axioms of the theory of Spec $\mathbb{R}(L)$ are validated in Max $\mathbb{R}(L)$ under this interpretation. There is therefore a map of locales

$$\text{Max } \mathbb{R}(L) \to \text{Spec } \mathbb{R}(L)$$

giving an isomorphism of Max $\mathbb{R}(L)$ with a sublocale of Spec $\mathbb{R}(L)$. It will be proved that there exists a map of locales

$$\text{Spec } \mathbb{R}(L) \to \text{Max } \mathbb{R}(L)$$

which provides a retraction of this embedding. It follows that:

(3.2) *For any locale L, the locale* Max $\mathbb{R}(L)$ *is compact,*

since Spec $\mathbb{R}(L)$ is always compact, which establishes the first assertion of (3.1). Intuitively, this generalises the classically observed fact that each prime ideal of $\mathbb{R}(L)$ is contained in a unique maximal ideal, obtained by taking the closure of the prime ideal. This observation motivates the description of the required map of locales, obtained by verifying that assigning to each $a \in A(q)$ of Max $\mathbb{R}(L)$ the proposition

$$\bigvee_{q'>q} (a:q') \in P$$

of Spec $\mathbb{R}(L)$ validates the axioms of the theory of Max $\mathbb{R}(L)$. In particular, the proposition $a \in P$ of Max $\mathbb{R}(L)$ maps to the disjunction $\bigvee_{q>0}(a:q) \in P$ of Spec $\mathbb{R}(L)$, in which $(a:q)$ denotes the cutdown of the continuous function a by the rational q.

It must now be shown that each axiom of Max $\mathbb{R}(L)$ determines in this way an entailment which is provable in the theory Spec $\mathbb{R}(L)$. One assumption which may be made throughout is that only those $a \in A(q)$ which arise from nonnegative functions $a \in \mathbb{R}(L)$ need be considered. Moreover, it may be remarked that if a, b are nonnegative with $a \le b$, then

$$(a:r) \in P \vdash b \in P \tag{3.3}$$

is provable in Spec $\mathbb{R}(L)$ whenever $r>0$. For in the locale L, one has

$$[b<r] \le [a<r] \le [(a:r)=0] \quad \text{and} \quad [r/2<b] \vee [b<r] = 1$$

since $r/2<r$. Therefore,

$$c = \begin{cases} (a:r)/b & \text{when } r/2<b, \\ 0 & \text{when } b<r \end{cases}$$

defines a continuous real function on L, which is bounded since b is bounded below when $r/2<b$ and which satisfies

$$(a:r) = bc.$$

Then, (3.3) is provable in Spec $\mathbb{R}(L)$ by (P4).

Now, the axioms of the theory of Max $\mathbb{R}(L)$ are shown to be satisfied in the locale Spec $\mathbb{R}(L)$ in the following way:

(A1): It must be proved in Spec $\mathbb{R}(L)$ that

$$true \vdash \bigvee_{r'>r} (1:r') \in P \quad \text{when } r<1.$$

But, given $r<1$, choose r' with $r<r'<1$. Then $(1:r')$ is the constant rational $1-r'$, which is invertible in $\mathbb{R}(L)$. Hence, $true \vdash (1:r') \in P$ by (P1) and (P4).

(A2): It must be proved in Spec $\mathbb{R}(L)$ that

$$\bigvee_{r'>r} (a:r') \in P \vdash false \quad \text{when } \|a\| \le r.$$

But, given $r'>r\geq \|a\|$ one has $|a|>r'$ in $\mathbb{R}(L)$. Then $|a|-r'<0$, so $(a:r')=0$. Thus $(a:r')\in P\vdash false$ by (P1).

(A3): It must be proved in Spec $\mathbb{R}(L)$ that

$$\bigvee_{t>r+s}(a+b:t)\in P\vdash \bigvee_{r'>r,s'>s}(a:r')\in P\vee(b:s')\in P.$$

Given $t>r+s$. choose t' with $r+s<t'<t$, and $r'>r, s'>s$ with $t'=r'+s'$. Then

$$((a+b)-(r'+s'))\vee 0\leq(a-r')\vee 0+(b-s')\vee 0$$

in $\mathbb{R}(L)$: that is, $(a+b:t')\leq(a:r')+(b:s')$. It follows by (3.3) that $(a+b:t)P\vdash ((a:r')+(b:s'))\in P$ is provable in Spec $\mathbb{R}(L)$ since $t>t'$. Hence,

$$(a+b:t)\in P\vdash(a:r')\in P\vee(b:s')\in P \quad \text{by (P3).}$$

(A4): It must be proved in Spec $\mathbb{R}(L)$ that

$$\bigvee_{r'>r,s'>s}(a:r')\in P\wedge(b:s')\in P\vdash \bigvee_{t>r^2+s^2}(a^2+b^2:t)\in P.$$

Given $r'>r, s'>s$, choose t and q such that $r^2+s^2<t<t+q<r'^2+s'^2$. Then

$$[\![(a^2+b^2:t)<q]\!]\leq[\![a^2+b^2<t+q<r'^2+s'^2]\!]\leq[\![a<r'\vee b<s']\!]$$
$$\leq[\![(a:r')=0\vee(b:s')=0]\!]\leq[\![(a:r')(b:s')=0]\!]$$

in the locale L. Moreover, $q/2<q$ implies that

$$[\![(a^2+b^2:t)>q/2]\!]\vee[\![(a^2+b^2:t)<q]\!]=1$$

in L. Therefore,

$$c=\begin{cases}(a:r')(b:s')/(a^2+b^2:t) & \text{when } (a^2+b^2:t)>q/2,\\ 0 & \text{when } (a^2+b^2:t)<q\end{cases}$$

defines a continuous real function on L, which is bounded since $(a^2+b^2:t)$ is bounded below when $(a^2+b^2:t)>q/2$ and which satisfies

$$(a:r')(b:s')=(a^2+b^2:t)c.$$

Then,

$$(a:r')\in P\wedge(b:s')\in P\vdash(a:r')(b:s')\in P \qquad \text{by (P4)}$$
$$\vdash(a^2+b^2:t)\in P \qquad \text{by (P4)}$$

from which the required result follows.

(A5): It must be proved in Spec $\mathbb{R}(L)$ that

$$\bigvee_{t>rs}(ab:t)\in P\vdash \bigvee_{r'>r,s'>s}(a:r')\in P\vee(b:s'\in P.$$

Given $t>rs$, choose $r'>r$, $s'>s$ with $t>r's'>rs$. Then,

$$(ab-r's')\vee 0=((a-r')b+(b-s')r')\vee 0\leq((a-r')b)\vee 0+((b-s')r')\vee 0$$
$$=b((a-r')\vee 0)+r'((b-s')\vee 0)=(a:r')b+(b:s')r'.$$

That is, $(ab:r's') \leq (a:r')b + (b:s')r'$. It follows by (3.3) that $(ab:t) \in P \vdash ((a:r')b + (b:s')r') \in P$ is provable in Spec $\mathbb{R}(L)$ since $t > r's'$. Hence,

$$(ab:t)P \vdash (a:r') \in P \vee (b:s') \in P \qquad \text{by (P3) and (P4).}$$

(A6): It must be proved in Spec $\mathbb{R}(L)$ that

$$\bigvee_{r'<r, s'>s} (a:r') \in P \wedge (b:s') \in P \vdash \bigvee_{t>rs} (ab:t) \in P.$$

Given $r' > r$, $s' > s$, choose t and q such that $rs < t < t+q < r's'$. Then

$$[\![(ab:t) < q]\!] \leq [\![ab < t+q < r's']\!] \leq [\![a < r' \vee b < s']\!]$$
$$\leq [\![(a:r') = 0 \vee (b:s') = 0]\!] \leq [\![(a:r')(b:s') = 0]\!]$$

in the locale L. Moreover, $q/2 < q$ implies that $[\![(ab:t) > q/2]\!] \vee [\![(ab:t) < q]\!] = 1$ in L. Therefore,

$$c = \begin{cases} (a:r')(b:s')/(ab:t) & \text{when } (ab:t) > q/2, \\ 0 & \text{when } (ab:t) < q \end{cases}$$

defines a continuous real function on L, which is bounded as before and which satisfies

$$(a:r')(b:s') = (ab:t)c.$$

Then,

$$(a:r') \in P \wedge (b:s') \in P \vdash (ab:t) \in P$$

is provable in Spec $\mathbb{R}(L)$ as before.

(A7): It must be proved in Spec $\mathbb{R}(L)$ that

$$\bigvee_{r'>r} (a:r') \in P \dashv\vdash \bigvee_{r<s} \bigvee_{s<s'} (a:s') \in P,$$

which is immediately clear.

The assignment therefore gives a model of the theory of Max $\mathbb{R}(L)$ in the locale Spec $\mathbb{R}(L)$, hence a map

$$\text{Spec } \mathbb{R}(L) \to \text{Max } \mathbb{R}(L)$$

of locales. The map is a retraction of the embedding of Max $\mathbb{R}(L)$ in Spec $\mathbb{R}(L)$, for given $a \in A(q)$ in the theory of Max $\mathbb{R}(L)$, it is interpreted in the locale Spec $\mathbb{R}(L)$ by $\bigvee_{q'>q}(a:q') \in P$. The canonical $\wedge\bigvee$ homomorphism from the lattice Spec $\mathbb{R}(L)$ to the lattice Max $\mathbb{R}(L)$ then maps this to $\bigvee_{q'>q}(a:q') \in P$, that is, to $\bigvee_{q'>q}(a:q') \in A(0)$. However, this is provably equivalent to $\bigvee_{q'>q} a \in A(q')$ in the theory of Max $\mathbb{R}(L)$ (by (2.5)), which in turn is provably equivalent to $a \in A(q)$ by (A7). The inverse image of the composite of the maps Max $\mathbb{R}(L) \to$ Spec $\mathbb{R}(L)$ and Spec $\mathbb{R}(L) \to$ Max $\mathbb{R}(L)$ of locales is therefore the identity, as required. It follows that the locale Max $\mathbb{R}(L)$ is compact, since it is a retract of the locale Spec $\mathbb{R}(L)$ which is compact. The compactness of Spec $\mathbb{R}(L)$ is an immediate consequence of its axiomatisation involving only *finite* disjunctions of propositions, making it the locale generated by a distributive lattice.

Finally, it will be shown that:

(3.4) *For any locale L, the locale* Max $\mathbb{R}(L)$ *is completely regular.*

For it has been proved that each element of the locale Max $\mathbb{R}(L)$ is expressible in the form $\bigvee a \in P$. Hence, it is enough to show that each $a \in P$ is the join of elements which are completely below it in the lattice Max $\mathbb{R}(L)$. However, $a \in P$ is equivalent to $\bigvee_{q>0} a \in A(q)$ (by (A7)), and it is asserted that

$$a \in A(q) \lhd\!\!\lhd a \in P$$

in the lattice Max $\mathbb{R}(L)$, for any $q>0$. For given $q>0$, the family $a \in A(kq/2^i)$, for $i=0,1,\dots$ and $k=0,1,\dots,2^i$ depending on i, provides an interpolation ranging (in fact downwards) between $a \in A(q)$ and $a \in P$, provided that it can be shown that

$$a \in A(kq/2^i) \rhd a \in A((k+1)q/2^i) \qquad (3.5)$$

for all appropriate i and k. However, this follows from the observation that for any $r<s$, it is provable in Max $\mathbb{R}(L)$ that:

$$a \in A(r) \vee \neg a \in A(s). \qquad (3.6)$$

For again it may be assumed that a is nonnegative. Then choose r', s' with $r<r'<s'<s$ and $q>0$ with $q<r'-r$, $q<s-s'$. Then $\vdash 1 \in A(r'/s')$. But

$$1 \in A(r'/s') \vdash a+q \in A(r') \vee 1/(a+q) \in A(1/s')$$

(by (A4)) since $a+q$ is invertible in $\mathbb{R}(L)$ on the assumption that $a \geq 0$. So

$$\vdash a+q \in A(r') \vee 1/(a+q) \in A(1/s').$$

It is asserted that in the first case we may deduce $a \in A(r)$, while in the second $\neg a \in A(s)$. For $a+q \in A(r') \vdash a \in A(r) \vee q \in A(r'-r)$, and $q \in A(r'-r) \vdash$ *false* since $q<r'-r$. Thus $a+q \in A(r') \vdash a \in A(r)$. Equally, suppose $a \in A(s)$. Then $a+q \in A(s') \vee q \in A(s-s')$. But $q \in A(s-s') \vdash$ *false* since $q<s-s'$. And thus $a \in A(s) \vdash a+q \in A(s')$. But in case $1/(a+q) \in A(1/s')$ we therefore have that $a \in A(s) \vdash 1 \in A(1)$ by (A6), whereas $1 \in A(1) \vdash$ *false*, by (A2). So $1/(a+q) \in A(1/s') \vdash \neg a \in A(s)$. Hence, the required disjunction proving (3.6). Observing that this states exactly that

$$a \in A(r) \rhd a \in A(s)$$

whenever $r<s$, this gives the assertion concerning the interpolation between $a \in A(q)$ and $a \in P$. The locale Max $\mathbb{R}(L)$ is therefore completely regular, which completes the proof of the main result.

The assignment to each locale L of the locale Max $\mathbb{R}(L)$ therefore gives the Stone-Čech compactification of the locale L, coinciding therefore with the constructions detailed elsewhere. In particular, the following corollary has also been proved:

Corollary. *For any locale L, the locale* Max $\mathbb{R}(L)$ *is a retract of the locale* Spec $\mathbb{R}(L)$.

Incidentally, it follows that the locale Max $\mathbb{R}(L)$ may equivalently be obtained by adjoining to the theory of Spec $\mathbb{R}(L)$ the axiom:

(P6) $$a \in P \vdash \bigvee_{q>0} (a:q) \in P.$$

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